1 Excercise 1.7.13

Consider the complete graph K_4 with four vertices; all vertices are connected by an edge to all other vertices. Suppose now we flip an unbiased coin for each edge. If heads comes up, we leave the edge where it is, if tails comes up, we remove the edge.

1.1 First Question

What is the probability that two given vertices are still connected after the removal of edges?

We note that when there are 4 or more edges, the graph is complete and as such any 2 given vertices are connected. When there are 0 edges, any 2 given vertices cannot be connected. When there is 1 edge, there is $\left(\frac{1}{2}\right)^6$ probability that the correct edge remained. When there are 2 edges, one can make a '2-step'-connection by connecting A to B via a third vertex (Where A and B are the two vertices who need to be connected). Since there are 4 vertices in total, this can be done in 2 ways. However, we can also make a direct connection using 2 edges. The first edge connects A and B directly, and the other edge can be randomly placed anywhere at one of the five remaining places. This gives us another 5 possible layouts using 2 edges, bringing the total to 7. The probability that exactly two edges remain is $\left(\frac{1}{2}\right)^6$, so the probability that there are exactly two edges which connect A and B is $7\left(\frac{1}{2}\right)^6$. Now we look at the problem when there are exactly three edges. When connecting from A to B with three edges it is possible to go from A to D to C to B, and from A to C to D to B. Apart from these two possibilities, it is possible that the three edges can form a 2-step connection, and that the third is put at some other random location. This greatly increases the total number of possibilities, and if it weren't enough, three edges can also make a direct connection and place the other two randomly somewhere else. Counting all these possibilities promises to be a little irritating, so perhaps it is better to look for ways where three edges are placed in such a manner that A and B are not connected. This indeed is a lot more fruitful; there are only two ways in which A and B are not connected. This is when you connect A to C to D to A, or B to C to D to B. Any other layout will cause a connection to B or, respectively, A, and thus connecting A to B (or B to A). Now we only need to look for the total number of ways we can put three edges on 6 places. This is where Newton comes in: $nCr(6,3) = \frac{6!}{3!(6-3)!} = 20$. So, $20 - 2 = 18$ is the number of right ways we can put the three edges, making the probability that three edges make a connection between A and B equal to $18\left(\frac{1}{2}\right)^6$. With edges higher than three we knew that A and B always are connected, so we need only to compute the total number of possibilities for each number of edges using Newton's formula. We get:

Number of edges	Probability
4	$15\left(\frac{1}{2}\right)^6$
5	$6\left(\frac{1}{2}\right)^6$
6	$\left(\frac{1}{2}\right)^6$

So the total probability that two vertices remain connected is:
$(1+7+18+15+6+1)\left(\frac{1}{2}\right)^6 = (48)\left(\frac{1}{64}\right) = \frac{3}{4}$

1.2 Second Question

What is the probability that the graph remains connected?

We note that a 4-vertex graph always is connected if the number of edges exceeds three. On the other hand, if this number is lower than 3, the graph never can be completely connected. So let us begin by setting up a table displaying the probability that any number of edges remain:

Number of edges	Probability
0	$\left(\frac{1}{2}\right)^6$
1	$6\left(\frac{1}{2}\right)^6$
2	$15\left(\frac{1}{2}\right)^6$
3	$20\left(\frac{1}{2}\right)^6$
4	$15\left(\frac{1}{2}\right)^6$
5	$6\left(\frac{1}{2}\right)^6$
6	$\left(\frac{1}{2}\right)^6$

Now we only have to determine how many times it goes right when there are three edges left. We saw, in the previous question, that for two given vertices there are 2 ways to connect them using a 3-step connection. We know that when one uses a 3-step connection in a 4-vertex graph to connect 2 given vertices, you have constructed a complete graph. We also know that a 4-vertex graph has six different vertex-pairs (AB, AC, AD, BC, BD, CD). So you can connect all 6 pairs in 2 different ways to get a comlete graph, and we conclude that there are 12 correct correct ways to organize 3 edges to get a complete graph. Now we can compile a table which displays the probability of a complete graph given the number of edges:

Number of edges	Probability for a complete graph
0	0
1	0
2	0
3	$12\left(\frac{1}{2}\right)^6$
4	$15\left(\frac{1}{2}\right)^6$
5	$6\left(\frac{1}{2}\right)^6$
6	$\left(\frac{1}{2}\right)^6$

Now add up the right column of this table to collect our answer: $34\left(\frac{1}{2}\right)^6 = \frac{17}{32}$

1.3 Third Question

What is the probability that a given vertex becomes isolated?

All vertices are initially connected to all three other vertices. So, for a vertex to become isolated, all three edges connected to that vertex need to be removed. For this to happen, the coinflips for each relevant edge have to come up with tails, and so the probability is:

$$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$$