Secretary's Problem

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May 9, 2004

Question 1

Show that the probability that a given envelope contains the correct letter is equal to $1/n$

When we have a given envelope, we can choose from n different letters to put into the envelope. Only one of these letters however is the correct one, so the probability to get a correct match naturally becomes $1/n$.

Question 2

Use question 1 to show that the expectation of X is equal to 1.

The random variable X represented the quantity of correctly matched letters and envelopes. Suppose we define $Y_i = 1$ as the probability that the i-th envelope was correctly matched with its letter, and $Y_i = 0$ otherwise. Then:

$$X = \sum_{i=1}^{n} Y_i$$

We know that $E(Y_i) = 1/n$ (from the previous question) and so:

$$E(X) = E(\sum_{i=1}^{n} Y_i) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} 1/n = n\frac{1}{n} = 1$$

Question 3

Show that the variance of X is also equal to 1.

We know that $\text{Var}(X) = E(X^2) - (E(X))^2$, and $(E(X))^2 = 1$, so we only need to compute $E(X^2)$. We could say that $X^2 = \sum_{i=1}^{n^2} Y_i = \frac{n^2}{n} = n$. So then we have $\text{Var}(X) = n - 1$...?

Question 4

Show that for all k,

$$P(X = k) \to \frac{e^{-1}}{k!},$$

as $n \to \infty$.

When we view the random variable X when n tends to infinity, we can look at X as a (infinite) binomial distribution. Binomial in the sense that when we draw an envelope, you can put the right letter in it, or the wrong letter. After putting any letter in it, we draw another envelope, and since $n \to \infty$, this process goes on forever. And as such, X becomes an Poisson-distribution with $\lambda = 1$ since $E(X) = 1$. We know that for any Poisson distribution:

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda},$$

and with $\lambda = 1$, $P(X = k) = \frac{e^{-1}}{k!}$. So indeed when n tends to go to infinity, the probability mass function closes in to $\frac{e^{-1}}{k!}$.