1 Excersice 5.12.30

Let X and Y be independent with a standard normal distribution. Show that X/Y has a Cauchy distribution.

First we find a joint distribution function of both X and Y. We know that both have a standard normal distribution:
$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-1/2x^2}$, en
$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-1/2y^2}$.

Also we know that:
$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{\infty} f_Y(y)dy = 1$.

So a joint distribution can be given by:
$f(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi} e^{-1/2(x^2+y^2)}$.

We can check that this is indeed a joint distribution of X and Y:
$\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = f_X(x) \int_{-\infty}^{\infty} f_Y(y)dy = f_X(x)$
$\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = f_Y(y) \int_{-\infty}^{\infty} f_X(x)dx = f_Y(y)$

We can now apply theorem 5.6.5 using this joint distribution function, and $g(x,y) = (x, x/y)$. We get
$g^{-1}(x,y) = (x, x/y)$, and
$J(g(x,y)) = \frac{dg_1}{dx}\frac{dg_2}{dy} - \frac{dg_2}{dx}\frac{dg_1}{dy} = 1 \cdot \frac{x}{y^2} - \frac{1}{y} \cdot 0 = \frac{x}{y^2}$.
Then we get:

$$\begin{aligned}
f_{X,Y}(x,y) &= f(g^{-1}(x,y))|J(g(x,y)|, \\
&= f(x, x/y)|\frac{x}{y^2}| \\
&= f(x, x/y)\frac{1}{y^2}|x| \\
&= \frac{1}{2\pi} e^{-1/2(x^2+(x/y)^2)} \frac{1}{y^2}|x| \\
&= \frac{1}{2\pi y^2} e^{-1/2x^2(1+\frac{1}{y^2})}|x|
\end{aligned}$$

Now we calculate the marginal distribution function $f_U(u)$, where $U = X/Y$:

$$\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_{X,Y}(x,u)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi y^2} e^{-1/2x^2(1+\frac{1}{u^2})}|x|dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi u^2}\int_{-\infty}^{\infty} e^{-1/2x^2(1+\frac{1}{u^2})}|x|dx \\
&= \frac{1}{2\pi u^2}(\int_{-\infty}^{o} -xe^{-1/2x^2(1+\frac{1}{u^2})}dx + \int_{0}^{\infty} xe^{-1/2x^2(1+\frac{1}{u^2})}dx \\
&= \frac{1}{2\pi u^2}(|\frac{1}{1+\frac{1}{u^2}}e^{-1/2x^2(1+\frac{1}{u^2})}|_{u=-\infty}^{0} + | - \frac{1}{1+\frac{1}{u^2}}e^{-1/2x^2(1+\frac{1}{u^2})}|_{u=0}^{\infty}) \\
&= \frac{1}{2\pi u^2}(2\frac{1}{1+\frac{1}{u^2}}) \\
&= \frac{1}{\pi u^2(1+\frac{1}{u^2})} \\
&= \frac{1}{\pi u^2 - \pi} \\
&= \frac{1}{\pi} \cdot \frac{1}{u^2+1}
\end{aligned}
$$

So, $f_U(u) = \frac{1}{\pi}\frac{1}{u^2+1}$ and we can conclude that $U = X/Y$ has a Cauchy distribution.