# Inleveropgave 8

Albert-Jan Yzelman (0216313)

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## Excersice 5.12.30

Let $X$ and $Y$ be independent exponentially distributed random variables with parameter 1. Let $Z = Y/X$.

### a)

Compute the distribution function of $(X, Z)$.

We begin with calculating the joint distribution function of $(X, Y)$:

$$\begin{aligned} f_X(x) &= e^{-x} \\ f_Y(y) &= e^{-y} \\ f_{X,Y}(x,y) &= e^{-(x+y)} \end{aligned}$$

We take $g(x, y) = (x, y/x)$ and calculate:

$$\begin{aligned} g^{-1}(x,z) &= (x, zx) \\ J(x,z) &= \frac{dg_1^{-1}}{dx} \cdot \frac{dg_2^{-1}}{dy} - \frac{dg_1^{-1}}{dy} \cdot \frac{dg_2^{-1}}{dx} \\ &= 1 \cdot x - 0 \\ &= x \\ f_{X,Z}(x,z) &= f_{X,Y}(g^{-1}(x,z))|J(x,z)| \\ &= f_{X,Y}(x, xz)|x| \\ &= xe^{-x(1+z)} \\ F_{X,Z}(x,z) &= \int_0^z \int_0^x f_{X,Y}(\xi,\eta) d\xi d\eta \\ &= \int_0^x \int_0^z \xi e^{-\xi(1+\eta)} d\eta d\xi \end{aligned}$$

$$
\begin{aligned}
&= \int_0^x |-e^{-\xi(1+\eta)}|_{\eta=0}^z d\xi \\
&= \int_0^x e^{-\xi} - e^{-\xi(1+z)} d\xi \\
&= |\frac{1}{1+z}e^{-\xi(1+z)} - e^{-\xi}|_{\xi=0}^x \\
&= \frac{1}{1+z}e^{-x(1+z)} - e^{-x} - \frac{1}{1+z} + 1
\end{aligned}
$$

## b)

Are $X$ and $Z$ independent?

If they are, then $F_{X,Z}(x,z) = F_X(x)F_Z(z)$. So lets calculate $F_X(x)$ and $F_Y(y)$:

$$
\begin{aligned}
f_Z(z) &= \int_0^\infty f_{X,Z}(x,z)dx \\
&= \int_0^\infty xe^{-x(1+z)}dx \\
&= |\frac{-1}{1+z}e^{-x(1+z)}(x+\frac{1}{1+z})|_{x=0}^\infty \\
&= \frac{1}{(1+z)^2} \\
F_Z(z) &= \int_0^z f_Z(\xi)d\xi \\
&= \int_0^z \frac{1}{(1+\xi)^2}d\xi \\
&= |\frac{-1}{(1+\xi)^3}|_{\xi=0}^z \\
&= 1 - \frac{1}{(1+z)^3} \\
F_X(x) &= \int_0^x f_X(\xi)d\xi \\
&= \int_0^x \xi e^{-\xi(1+z)}d\xi \\
&= |-\frac{1}{1+z}e^{-\xi(1+z)}(\xi - \frac{1}{1+z})|_{\xi=0}^x \\
&= -\frac{1}{1+z}e^{-x(1+z)}(x - \frac{1}{1+z}) + \frac{1}{(1+z)^2}
\end{aligned}
$$

We can immediatly see that $F_X(x)F_Z(z)$ is not equal to $F_{X,Z}(x,z)$, so $X$ and $Z$ are not independant.

## c)

Compute $E(Z|X = x)$.

First we calculate the conditional density function:

$$
\begin{aligned}
f_{Z|X}(Z|X) &= \frac{f_{X,Z}(x,z)}{f_X(x)} \\
&= \frac{xe^{-x(1+z)}}{e^{-x}} \\
&= xe^{-xz}
\end{aligned}
$$

Now we calculate the asked expection:

$$
\begin{aligned}
E(Z|X = x) &= \int_0^\infty z f_{Z|X}(z,x) dz \\
&= \int_0^\infty zxe^{-xz} dz \\
&= | - e^{-xz}(z - \frac{1}{x})|_{z=0}^{\infty} \\
&= 0 - (-\frac{1}{x}) \\
&= \frac{1}{x}
\end{aligned}
$$